Anderson (De)localization in a Complex Plasma Crystal at Weak Disorder

Kyle Busse, Constanze D. Liaw, Lorin S. Matthews, and Truell Hyde

***Abstract*—The Anderson Localization Conjecture (that random disorder of any strength causes electron localization within an ordered medium) is a well-known field of research in solid state physics. The Conjecture, which has been proven mathematically for strong disorder in both the 2– and 3–dimensional cases, has been challenged by Liaw for the 3–dimensional case at weak disorder via numerical findings that are based to a further extent on a mathematically rigorous treatment. We seek to treat the case of a 2–dimensional case for the physical system of plasma a plasma crystal. To do so, we utilize a numerical program to simulate a complex plasma crystal, introducing weak disorder using a thermal bath (a setup easily realized in the laboratory setting). A perturbation is introduced into the crystal by manually imparting a large nonzero velocity vector on one of the dust particles and the spread of kinetic energy throughout the crystal is observed. We couple this numerical simulation with the mathematical method developed by Liaw to show that the Anderson Localization Conjecture is false in a complex plasma crystal at weak disorder.**

***Index Terms*—Dusty plasmas, plasma simulation**

I. Introduction

THE Anderson Localization Conjecture and the phenomenon of Anderson localization have been widely explored by the mathematics and physics communities over the last six decades. The conjecture has been proved mathematically for all dimensions at strong disorder [1]–[4] and has been shown to be false for the three-dimensional case at weak disorder [5]. Quite recently, numerical evidence has been presented that the conjecture does not hold for the two-dimensional case at weak disorder [5].

Previous studies of the conjecture for the 2–dimensional case at weak disorder have arrived at contradicting claims (largely as a result of problems arising from boundary effects) arrived at via unsatisfactory mathematical methods. In order to study the conjecture with more mathematical rigor, Liaw introduced a new method in 2013 based on spectral and operator theory that obviates the necessity for an acceptable approach to research in this direction [6]. One of the extraordinary advantages of this method is its versatility. It can be applied to any ordered system such as colloids, complex plasma crystals, and atomic lattices. In this sense, the long-term goals of this work consist of an analysis of the conductive properties of materials in terms of the system parameters. A prime example could the 2–dimensional material graphene (a crystal-like material with hexagonal structure) which has enjoyed much attention, especially since groundbreaking experiments regarding the two-dimensional material graphene was the topic of the 2009 Physics Nobel prize (awarded to Geim and Novoselov). The fact that graphene is one of the materials used for solar cells provides further motivation for our work. Extended states allow for more efficient production of electricity, whereas localized states make the cell less efficient.

In order to apply this mathematical approach, it was necessary to find a suitably realistic simulation of an ordered 2–dimensional system at weak disorder. An N-body program developed by Matthews, et al. [7], [8] designed to simulate a complex plasma crystal meets this requirement. The simulated complex plasma crystal is nearly ideal for the purpose of our study – an equilibrium state is achieved in which the complex plasma crystal is of nearly perfect hexagonal crystalline structure (particles are at near–zero velocity) and weak disorder is introduced into the system via a thermal bath. The program provides all necessary physical data about the system, including the position and velocity of each particle at each time-step. A perturbation is introduced into the system by giving a single particle a non-zero velocity vector. Then the kinetic energy of each particle at each time step is calculated and its distribution throughout the crystal is examined.

By applying Liaw's mathematical approach to the bulk distribution of kinetic energy in the crystal, we observe delocalization within the system, thereby showing the falsity of the Conjecture for this 2–dimensional system. By combining our rigorous mathematical treatment with a physically realistic simulation, we propose a new direction of study to the field of Anderson Localization in which the rigorous truth is yet to be found and has been sought after for 50+ years by the physics and mathematics communities.

II. Numerical Simulation

The numerical simulation chosen for this analysis has been used extensively to model complex plasma systems and their behavior [7], [8]. This code simulates an arbitrary N number of particles within a complex plasma crystal, consisting of

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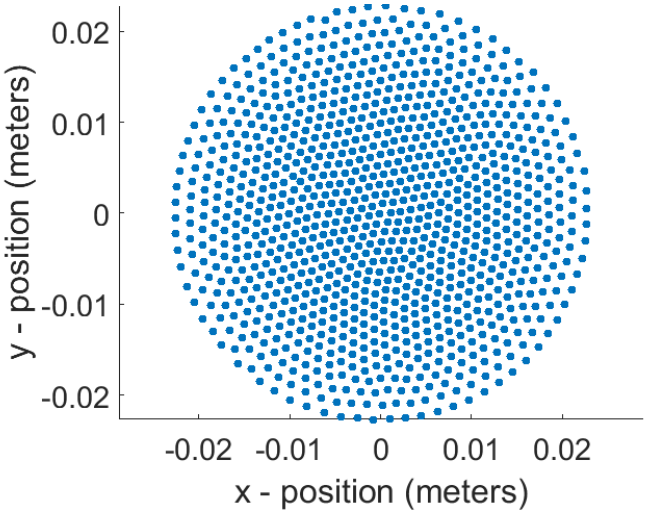


Fig. 1. A visualization of the top view of a simulated dusty plasma crystal using the box tree program

micrometer sized solid particles electrostatically confined in a plasma environment. The program itself takes as input a set of initial conditions, giving among other things, the charge, mass, radius, and position and velocity vectors for each particle. Note that in this simulation, the particles are treated as having constant mass, constant charge, and constant radius. This is not to the detriment of the ability for this system to be realized in the lab as the dust particles purchased for use in the lab are monodisperse with small deviation in size, and the charges are large enough so that fluctuations are negligible. As output, the program gives the position and velocity vectors of each particle at each time step. As reference, the particles bear a mass of 7.7×10^{-13} kg, a charge of -3.2×10^{-15} C, and a radius of 4.9×10^{-6} m.

A. Forces Acting, and Equilibrium Conditions

The simulated complex plasma crystal is strictly controlled, as are complex plasma crystals generated in the lab. The particles in the crystal are assumed to interact with one another through a Yukawa potential

$$V_{\text{Yukawa}} = \sum_{k \neq i} \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_{ik}} e^{-\lambda / r_{ik}}. \tag{1}$$

where λ denotes the Debye length of the plasma (kept at 300×10^{-6} m), q_i denotes the charge of particle i, r_{ij} denotes the radial distance between particles i and j, and ε_0 denotes the permittivity of free space.

The crystal itself is formed and confined by a Hooke's law type force, defined for each particle at a radial distance r from the origin (center of the plasma) by

$$F_{\text{conf}} = m\frac{d^2 r}{dt^2} = -kr. \tag{2}$$

Additionally, the particles in the system are acted upon by a drag force proportional to their velocity, with drag coefficients in the x, y, and z directions given by c_x, c_y, and c_z, respectively. We denote this drag force by

$$\vec{F}_{\text{drag}} = -\gamma \vec{v}. \tag{3}$$

In order to achieve a crystalline equilibrium state comprised of particles of near-zero velocity vectors, N particles are randomly placed into the system and allowed to reach equilibrium under a large drag force. The system was determined to be at equilibrium when the total kinetic energy of the system was very close to zero, at which point the average magnitude of the velocity of each particle was on the order of 10^{-7} m/s. Once equilibrium was achieved, the drag force was reduced and the only parameters varied were the magnitude of the random disorder (the thermal bath, c) present in the system, and time t.

B. Choice of Time Step

In order to resolve the true behavior of the motion, a proper time step had to be chosen. The period of F_{conf} was used as a rough estimate for the proper time scale, yielding $T = 2\pi/\omega = 2\pi/\sqrt{4500} \approx 0.1$ seconds. The time step was then chosen to be 0.0001 seconds with a termination time of 0.1 seconds (therefore, we obtain 1000 data files). This time step was in fact sufficiently small to resolve relevant change in the system, but was so small that repetitive data was recorded. This data is discussed along with the mathematical methodology in the pertinent section.

C. Type and Size of Perturbation

There were two primary options for introducing a perturbation into the system: giving a particle a large non–zero velocity and giving a particle a new position. In order to remain as close as possible to a realistic experiment, we decided to use the former. The particle chosen to be given this perturbation was of little importance to mathematical analysis of the system – indeed the mathematics sees no difference between a perturbation of one particle and another. Therefore, we chose a particle to perturb that was in a "perfectly" hexagonal crystalline region of the crystal (that is to say, away from defects). By choosing a particle in this way, we were assured to see the true nearest-neighbor interactions due to the perturbation as we would expect to see for a perfectly ordered crystal.

The matter of picking a size of velocity to give a particle was one of trial. Initially, the velocity given to the chosen particle was $v_{\text{pert}} = \langle 1.0 \times 10^{-3}, 1.0 \times 10^{-3}, 0 \rangle$ m/s. This proved to be too large, however, as it caused the perturbed particle to shift out of its original position. In doing this, it therefore introduced new "perturbations" which caused undesired kinetic energy to enter the system and altered the actual crystalline structure of the system. We decreased the magnitude of the perturbation, and found that one of $v_{\text{pert}} = \langle -1.0 \times 10^{-4}, -1.0 \times 10^{-4}, 0 \rangle$ m/s produced desirable results.

D. Implementation of Disorder

Although our mathematical approach is originally suited to analyze Anderson–type Hamiltonian operators with random

component distributed according to a uniform random distribution, we achieve our randomness by employing the random number generator introduced as "gasdev" in Numerical Recipes in C, which returns a normally distributed deviate with zero mean and unit variance [9], [10]. Using this generator, we introduced the thermal bath which acts to apply weak disorder to our system.

The thermal bath introduces disorder into the complex plasma system based on the principles of Brownian motion and the Langevin equations of motion for a Brownian particle of mass m. Consider such equations for a Brownian particle in one dimension, with position given as $x(t)$:

$$\begin{aligned} \frac{dx(t)}{dt} &= v(t) \\ \frac{dv(t)}{dt} &= -\frac{\gamma}{m}v(t) + \frac{1}{m}\xi(t). \end{aligned} \tag{4}$$

The Brownian particle experiences a drag force proportional to its velocity characterized by the drag coefficient γ in addition to a random force $\xi(t)$ subject to the constraint that the mean $\langle \xi(t) \rangle_\xi = 0$ and that the variance $\langle \xi(t_1)\xi(t_2) \rangle = g\delta(t_1 - t_2)$, where g denotes the strength of the force $\xi(t)$.

The disorderly component of our numerical model may be thought of as an Ornstein–Uhlenbeck process [11]. We may then appeal to the fluctuation–dissipation theorem to define

$$g = 2\gamma k_B T$$

where k_B is Boltzmann's constant and T is the thermodynamic temperature of the system.

The particles in laboratory plasma system experiments are in the Epstein drag regime (that is, the mean free path of the plasma particles is greater than the radius, a, of the dust particles), and γ (the drag coefficient) is given by

$$\gamma = \delta \frac{4\pi}{3} a^2 N m_{\mathrm{n}} \sqrt{\frac{8k_B}{\pi m_n}}. \tag{5}$$

III. Mathematical Approach

One of the key novel characteristics of our research is the use of Liaw's rigorous mathematical approach to studying the Anderson Localization conjecture and the Anderson–type Hamiltonian. Consider such an operator given by

$$H_\omega = -\triangle + \sum_{i \in \mathbb{Z}^2} \omega_i \langle \cdot, \delta_i \rangle \delta_i$$

acting on some unit vector u. By $\triangle$ we denote the Laplacian operator (i.e. ∇^2) and by ω_i we mean a random number distributed according to a uniform random distribution bounded by the level of disorder. The forward orbit of the operator acting on $u = \delta_1$ is observed as the set of vectors

$$\left\{ \delta_0, H_\omega \delta_0, H_\omega^2 \delta_0, \ldots, H_\omega^n \delta_0 \right\}. \tag{6}$$

These vectors are then orthonormalized using the classical Gram–Schmidt algorithm to obtain the set of vectors

$$\{ x_0, x_1, x_2, \ldots, x_n \}. \tag{7}$$

Then, the distance D_c^n is then defined to be

$$D_c^n := \mathrm{dist}\left(\mathrm{span}\left\{ x_0, x_1, \ldots, x_n \right\}, u \right).$$

In her work, Liaw gives an explicit formula for this distance:

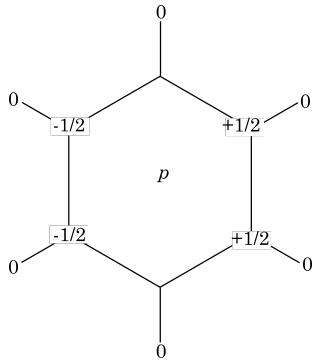
$$D_c^n = \sqrt{1 - \sum_{i=1}^{n} \langle x_i, \delta_1 \rangle^2} \tag{8}$$

and further proves that, for a given system, we lack Localization if and only if

$$\lim_{n \to \infty} D_c^n > 0. \tag{9}$$

A. Choice of the Unit Vector u

The unit vector $u = \delta_1$ was chosen to be asymmetrical about the perturbed particle. That is, for perturbed particle p, we have



Such an asymmetry is desirable. Without disorder in the system, the perturbation evolves symmetrically about the perturbed particle p. So we expect that for small disorder the states are close to symmetric. Therefore, an asymmetric vector δ_1 likely yields positive distance D^{n_0}.

B. Observed Characteristic of the System

The primary work to be done in applying Liaw's approach to the simulated complex plasma crystal is in calculating the vectors (3). This consists in observing the time evolution of some characteristic of the system that gauges the amount to which the system is affected by the perturbation. The characteristic eventually settled upon was the absolute kinetic energy of each particle in the system. To this end, at each time step, the kinetic energy of each particle was calculated as $\mathrm{KE} = 1/2m||v||^2$. These values were recorded in column vectors, call them "kinetic energy vectors," sorted from top to bottom by the distance from the perturbation (i.e. the first entry of each vector was the perturbed particle itself).

C. Linear Dependencies

As a consequence of the small time–step used, there were some kinetic energy vectors that were repeats of one another. That's to say that it occurred sometimes that the kinetic energy vector for a given time step was the same as the one directly afterward. This becomes a problem when we apply

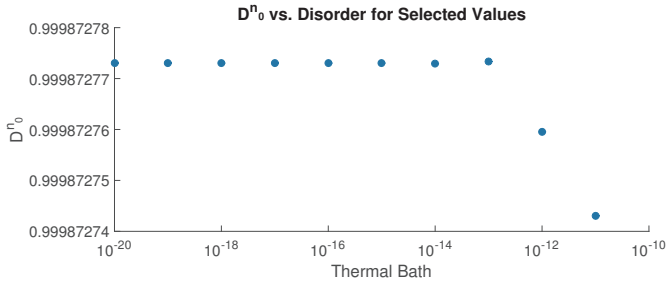


Fig. 2. Scatter plot of D^{n_0} against Level of Thermal Bath

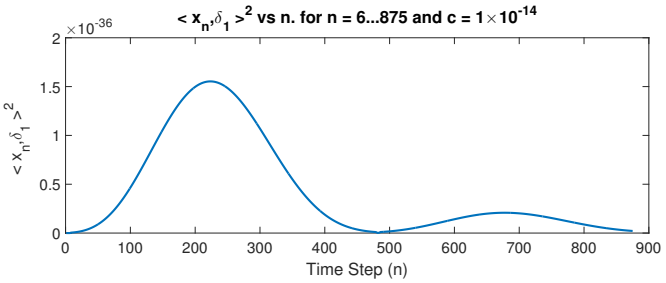


Fig. 3. Plot of $\langle x_n, \delta_0 \rangle$ against n showing the non-zero difference between D_c^n and D_c^{n+1}

TABLE I
CALCULATED DISTANCES AT VARIOUS VALUES OF DISORDER

Thermal Bath	D^{n_0}
1.0×10^{-8}	0.999869647862758
1.0×10^{-9}	0.999876669513230
1.0×10^{-10}	0.999871157619804
1.0×10^{-11}	0.999872746049918
1.0×10^{-12}	0.999872779121354
1.0×10^{-13}	0.999872808605408
1.0×10^{-14}	0.999872806111855
1.0×10^{-15}	0.999872806101277
1.0×10^{-16}	0.999872806101548
1.0×10^{-17}	0.999872806100705
1.0×10^{-18}	0.999872806100822
1.0×10^{-19}	0.999872806100822
1.0×10^{-20}	0.999872806100822

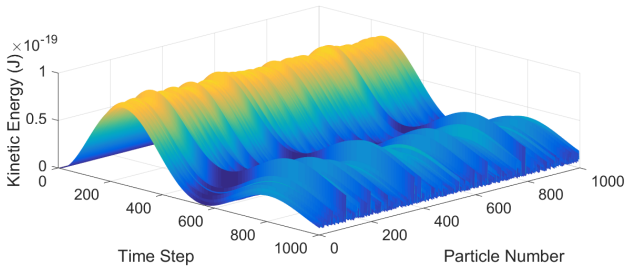


Fig. 4. A surface plot showing the kinetic energy of the system evolving over time

the Gram–Schmidt orthonormalization algorithm to the set of vectors because the set is clearly linearly dependent. To remedy this, we simply disposed of the kinetic energy vectors that were linearly dependent within the set. From the set of 1000 kinetic energy vectors, at most 129 were disposed of. For ease of notation, we shall mean by n_0 the number of linearly independent "kinetic energy" vectors at disorder c.

Note well that we haven't compromised any results by throwing out these vectors. By its nature, the way the distance is calculated using formula (6) would not permit linearly dependent vectors to affect D^n. From a physical standpoint, we are adjusting the sampling rate to only compute physically relevant data.

IV. Results

Following orthonormalization, the distances D^n were calculated for $n = 1$ to the total number of kinetic energy vectors (n_0). The distance D^n was then plotted against n in order to visualize its limit as n grew. Such plots clearly indicate that the limit for D^n is never negative and is indeed convergent. Therefore, for each value of weak disorder chosen, delocalization was observed.

Upon initial examination of the data D_c^n for all chosen values of c, it appeared that after $n = 1$ through $n = 5$, $D_c^n = D_c^{n-1}$. In other words, after the first five values of n, D^n appeared constant. This kind of homogeneity would be undesirable in the data. However, upon more careful examination, it was discovered that indeed the distance function *was* changing, albeit on a scale far smaller than that on which the distance itself was plotted. In order to visualize this difference, we have plotted the term $\langle x_n, \delta_1 \rangle^2$ against n. Because including the initial values for this inner product (which cause the large drop observed in D_c^n at the beginning of the graph) would skew the scale undesirably for visualization purposes, the first five values of n have been thrown out.

V. Conclusions and Future Work

By merging our rigorous mathematical technique with a realistic physical simulation of a complex plasma crystal, we have shown that such a two–dimensional system at weak disorder (introduced via a thermal bath) exhibits localization. Therefore, the Anderson Localization Conjecture has shown to be false in this situation.

As this is a new direction in the research of Anderson Localization, there are a large number of potential directions for work to evolve from here. In order to observe how D^{n_0} varied with the level of disorder, we created a dot plot showing these data (see Fig. 2). It was observed that D^{n_0} remains stable from $c = 1.0 \times 10^{-20}$ through $c = 1.0 \times 10^{-13}$, at which point we see a marked decrease. Indeed, for values of c less than 1×10^{-13}, instability is observed in such a dot plot. It is desirable, therefore, to come up with physical or mathematical rationale for this instability. We hypothesise that either the complex plasma crystal itself becomes unstable, or that we actually need several trials, and then to average over the trials to obtain smooth data. Ideally, we will be able to find a smooth "trend" among these data.

Another direction in which this work will evolve is towards answering questions about the effects of periodic boundary conditions on Localization within ordered media. We would like to understand how crystals of variable size respond to periodic boundary effects in general. Ideally, it would be expected that our mathematical model would be unfazed by

periodic boundary conditions – that's to say results obtained using a similar method to ours on crystals of variable size and periodic boundary conditions would still imply Localization or lack thereof.

We would like to perform similar tests as those demonstrated in this paper to determine the behavior of the crystal at different sizes of the perturbation. According to our mathematical mode, altering the size of the perturbation (not making it so large that we get changes in position as discussed in the pertinent section) ought not have any effect on D^{n_0}.

Additionally, we would like to develop a correlation between the level of disorder and the number of linearly independent "kinetic energy vectors" n_0.

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Together with her husband, Joe Kuehl, she has two wonderful children: Joanna (2006) and Calvin (2010). Dr. Liaw likes to learn and teach new, interesting things every day. She plays table tennis and loves downhill skiing.

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